# RATIONALIZATION OF GENERAL FORMULAS FOR ANGLE FACTORS

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Inzhenerno-Fizicheskii Zhurnal, Vol. 13, No. 4, pp. 482–488, 1967

UDC 536.2

General formulas are derived for the average angle factors of volumetric and surface zones involving only a single fourfold integral, and these formulas are therefore the most rational for numerical solutions.

Angle and radiation factors are found in approximation of integral radiant-energy transport equations by a system of algebraic equations [1]; they are used extensively in engineering calculations and they occupy a dominant position in zonal calculation; they are the subject of a special literature that is by no means complete. If the symbols V and F are used to denote, respectively, the volume and surfaces of the system, the average V-V and V-F coefficients are, in general, expressed in the form of sixfold and fivefold integrals. By means of algebraic relationships these can always be expressed in terms of coefficients of the F-F type, i.e., in terms of fourfold integrals [2]. In this case, there will be from two to four such integrals, calculated independently, in the formula. Here we have derived an arbitrary type of coefficient involving only a single fourfold integral. The rationalized formulas assume a new meaning—the meaning of a set or the density of a set of beams, linking the elements of the zones, at some arbitrary cross section. The angle-factor formulas for two surface zones cannot be simplified and are therefore not considered.

The attenuation (absorption and dissipation) capacity of the zonal segment $i(a_i)$ becomes important. Figure 1 shows an arbitrary surface F, a number of volumetric zones on either side, and a limiting system of surfaces $F_i$ and $F_j$. The ray passes through element dF. If it begins in this element, the quantity $a$ for the individual volumetric zones has the form

$$\begin{gathered} a_1 = 1 - \exp(-\tau_1), \\ a_2 = \exp(-\tau_1)\left(1 - \exp(-\tau_2)\right), \\ a_3 = \exp\left(-(\tau_1+\tau_2)\right)\left(1-\exp(-\tau_3)\right), \\ \cdots\cdots\cdots\cdots \\ a_i = \exp\left(-(\tau_1+\tau_2+\ldots\tau_{i-1})\right)\left(1-\exp(-\tau_i)\right), \quad (1) \\ a_b = 1 - \exp(-\tau_b), \\ a_c = \exp(-\tau_b)\left(1-\exp(-\tau_c)\right), \\ \cdots\cdots\cdots \end{gathered}$$

The quantity $a_i$ defines the probability that the quantum energy emitted at point dF is attenuated (absorbed or dissipated) in some direction on the segment $\tau_i$ of zone i. With the Kirchhoff law valid, the quantities $a_i$ serve also as characteristics of radiation for zone i passing through element dF

$$I = \frac{\alpha_i}{k_i} B_i a_i.$$

If we do not take into consideration all of the rays which reach point dF, but only those which continue

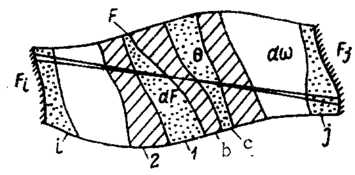


Fig. 1. Scheme of a chamber with surface ($F_i$, $F_j$) and volumetric zones (1, 2, ..., i, b, c, ..., j).

to zone j and are attenuated by that zone, their intensity at point dF is given by

$$I = \frac{\alpha_i}{k_i} B_i a_i a_j. \tag{2}$$

For the rays reaching zone $F_j$ through element dF

$$I = \frac{\alpha_i}{k_i} B_i a_i \left[1 - (a_b + a_c + \ldots + a_j)\right], \tag{3}$$

etc.

The densities of the hemispherical flows at point dF are expressed identically

$$q = \int_{2\pi} I \cos\theta \, d\omega. \tag{4}$$

Here as before [1, 3], angle factors are uniformly denoted and defined for any zonal pair. An angle factor multiplied by the optical constant of the object of irradiation is dimensionless and defines the probability that the quantum of energy emitted by the zone of the first index will reach the zone of the second index directly (without interaction with elements of the system) and will be absorbed there, if the optical constant is the absorption coefficient. The probability that the quantum of energy will be emitted by the element dF within the solid angle $d\omega$ through an angle $\theta$ is equal to $\cos\theta \, d\omega/\pi$. The solid-angle integral of the product of this quantity and $a_i$ defines the local angle factor from point dF to zone i, multiplied by the attenuation factor $k_i$,

$$\xi_{dF,\,i} k_i = \int_{2\pi} a_i \frac{\cos\theta \, d\omega}{\pi}. \tag{5}$$

The quantity $\xi_{dF,\,i} k_i$ (when $\beta = 0$) is the local absorption capacity of the volume [4]. The average angle factor from surface F to zone i is defined by

$$\varphi_{F,\,i} = \frac{1}{F} \int_F \xi_{dF,\,i} \, dF. \tag{6}$$

The average angle factor from zone i to surface F is defined according to the reciprocity relationship [1]

$$F\,\varphi_{F,i} = 4V_i\varphi_{i,F}, \qquad (7)$$

and thus according to (5)–(7),

$$\varphi_{i,F} = \frac{1}{4\pi V_i k_i}\int_F dF \int_{2\pi} a_i \cos\theta\, d\omega. \qquad (8)$$

Volumetric zone j is situated on the other side of surface F. From (8) we find the angle factor from zone i to zone j, multiplied by $k_j$, if the absorption factor for element dF of the integrand (unity) is replaced by $a_j$

$$\varphi_{ij}k_j = \frac{1}{4\pi V_i k_i}\int_F dF \int_{2\pi} a_i a_j \cos\theta\, d\omega; \qquad (9)$$

$\varphi_{ij}k_j$ defines the probability that the quantum of energy emitted by zone i will reach zone j directly and will be attenuated by that zone. Now it is not difficult to derive the formulas for other coefficients. For example, the coefficient from volumetric zone 1 to closed shell F bounding this zone is expressed by formula (8), in which i should be replaced by the subscript 1. We define the coefficient for zone 1 from the relationship of the latter closing on itself:

$$\varphi_{11}k_1 = 1 - \varphi_{1F} = 1 - \frac{1}{4\pi V_1 k_1}\oint_F dF \int_{2\pi} a_1 \cos\theta\, d\omega. \qquad (10)$$

In special cases the integral can be simplified. For example, for a cube it is enough to carry out the integration over a single face. The angle factor from zone i to surface $F_j$ is given by

$$\varphi_{iF_j} = \frac{1}{4\pi V_i k_i}\int_F dF \int_{2\pi} a_i \times$$

$$\times [1 - (a_b + a_c + \ldots + a_j)]\cos\theta\, d\omega. \qquad (11)$$

Here we have used (3). It must be stipulated that F intersect the entire set of rays linking the element of the zonal pairs.

The quantity $\mu$ defined from $\mu = q/Q_i$, may be referred to as the density of the set of rays at point dF, proceeding from zone i to the zone beyond element dF. Here $Q_i = 4V_i\alpha_i\pi B_i$ is the intensity of the self-radiation of zone i. Considering (4),

$$\mu = \frac{1}{4\pi V_i a_i B_i}\int_{2\pi} I\cos\theta\, d\omega.$$

When the radiation passes to volumetric zone j (Fig. 1), the quantity I is defined by (2) and then

$$\mu_{ij} = \frac{1}{4\pi V_i k_i}\int_{2\pi} a_i a_j \cos\theta\, d\omega.$$

The set of rays passing through F is found by integration:

$$M_{ij} = \int_F \mu\, dF = \frac{1}{4\pi V_i k_i}\int_F dF \int_{2\pi} a_i a_j \cos\theta\, d\omega. \qquad (12)$$

The right-hand members of (9) and (12) coincide so that $\varphi_{ij}k_j = M_{ij}$. The derivative of $\varphi_{ij}k_j$ with respect to F yields the set density $\mu_{ij}$. Thus the new, general formula (9) and similar formulas assume a simple meaning.

Formulas (8)–(11) and those similar to these are easily generalized to the case of the effective radiation of zone i with an arbitrary indicatrix, even for selective radiation. With selective radiation the formulas retain their form, but the quantities $a_i$ and $a_j$ must have different expressions that are more complex than (1). This problem requires special treatment.

**Examples.** Figure 2 shows a system of two volumetric zones i and j with spherical or cylindrical symmetry. The subscripts 1 and 2 denote the shells of the zones. Surface 3 is a part of surface 2, supported by the tangents to the latter, drawn from point $dF_1$. In view of symmetry, integration over F (here $F_2$) is dropped. Formulas (8)–(11) are simplified:

$$\varphi_{iF_2} = \frac{F_2}{4\pi V_i k_i}\int_{2\pi} a_i \cos\theta\, d\omega,$$

$$\varphi_{ij}k_j = \frac{F_2}{4\pi V_i k_i}\int_{2\pi} a_i a_j \cos\theta\, d\omega,$$

$$\varphi_{ii}k_i = 1 - \frac{F_2}{4\pi V_i k_i}\int_{2\pi} a_i \cos\theta\, d\omega,$$

$$\varphi_{iF_1} = \frac{F_2}{4\pi V_i k_i}\int_{2\pi} a_i(1 - a_j)\cos\theta\, d\omega,$$

where

$$a_i = 1 - \exp(-\tau_i), \quad a_j = 1 - \exp(-\tau_j).$$

In the following we use the radii multiplied by the attenuation factor, which is assumed to be constant, i.e., dimensionless radii r and R.

For spherical symmetry $\tau_i = 2r\cos\theta$, $\tau_j = (R^2 - r^2\sin^2\theta)^{1/2} - r\cos\theta$, $d\omega = 2\pi\sin\theta d\theta$, $F_2/V_ik_i = 3/r$.

For cylindrical symmetry

$$\tau_i = 2r\frac{\cos\varepsilon}{\sin\gamma},$$

$$\tau_j = \frac{\sqrt{R^2 - r^2\sin^2\varepsilon} - r\cos\varepsilon}{\sin\gamma} = r\frac{\sqrt{\delta^2 - \sin^2\varepsilon} - \cos\varepsilon}{\sin\gamma},$$

$$d\omega = \sin\gamma\, d\gamma\, d\varepsilon, \ \cos\theta = \sin\gamma\cos\varepsilon, \ F_2/V_ik_i = 2/r.$$

References [5, 6] have been published on the subject of angle factors for a spherical system. Reference [5], in which some of the quantities have been tabulated, is of greater interest here. But unlike this last reference, we have carried out the integration to the end for all of the coefficients. This yielded

$$\varphi_{iF_2} = \frac{3}{8r^3}[f(2r) - 1 + 2r^2], \quad \varphi_{ii}k_i = 1 - \varphi_{iF_2},$$

$$\varphi_{ij}k_j = \frac{3}{8r^3}[2r^2 + f(2r) - 1 + f(R-r) - f(R+r) + (R-r)^2 E_3(R+r) - (R+r)^2 E_3(R-r)],$$

$$\varphi_{iF_1} = \frac{3}{8r^3}[f(R+r) - f(R-r) + (R+r)^2 E_3(R-r) - (R-r)^2 E_3(R+r)].$$

To obtain a complete solution, below we write the angle factors for the surface zones

$$\varphi_{F_1F_1} = \frac{1}{2R^2}[1 - f(2R)],$$

$$\varphi_{F_2F_1} = \frac{1}{2r^2}[f(\sqrt{R^2 - r^2}) - f(R-r) + (R+r)^2 E_3(R-r) - (R^2 - r^2) E_3(\sqrt{R^2 - r^2})].$$

However, if shell $F_2$ is opaque,

$$\varphi_{F_1F_1} = \frac{1}{2R^2}[1 - f(2\sqrt{R^2 - r^2})].$$

For surfaces $F_1$ and $F_3$ (without surface $F_2$)

$$\varphi_{F_1F_3} = \frac{1}{2R^2}[f(\sqrt{R^2 - r^2}) - f(R+r) + (R-r)^2 E_3(R+r) - (R^2 - r^2) E_3(\sqrt{R^2 - r^2})].$$

Table 1

Angle factors from a sphere (dimensionless radius r) to spherical shell (dimensionless radius δr). The zones are separated by a spherical layer of a medium exhibiting thickness (δ − 1)r. (Here and in Tables 2 and 3 the numbers after the decimal point are shown.)

| r | δ | | | |
|---|---|---|---|---|
| | 1.04 | 1.1 | 2 | 4 |
| 0.1 | 92354 | 91619 | 82813 | 67436 |
| 0.5 | 68687 | 65962 | 39736 | 14235 |
| 1.0 | 49657 | 45745 | 16563 | 02166 |
| 2 | 29370 | 24861 | 32565 | 00054 |
| 5 | 10690 | 07057 | 00042 | 0 |

The functions $E_n(x)$ are used extensively in various characteristics of radiation from a flat layer. For example, $2E_3(x)$ defines the fraction of diffusely incident radiation passed by a layer without interaction, and this goes under the name "transmission" [7]. This

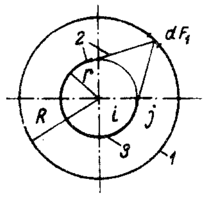


Fig. 2. Section of simplest system of concentric and coaxial zones.

same function with denotation $s_1(x)$ is used in characteristics of cylindrical systems [8]. Now the function $E_3(x)$ is found in the characteristics of spherical systems. Detailed tables are found in the literature only for functions $E_1(x)$. From the recursion formula $(n-1)E_n(x) = \exp(-x) - xE_{n-1}(x)$ we can define the functions $E_n$ for any n in terms of $E_1(x)$. However, it is simpler to use approximate formula (9)

$$E_3(x) = 0.2645\exp(-1.1612x) + 0.2355\exp(-2.942x).$$

We tabulated the coefficient $\varphi_{iF_1}$ for spherical symmetry, and $\varphi_{F_2F_1}$ and $\varphi_{ij}$ for cylindrical symmetry. Finally they assume the form

$$\varphi_{ij}k_j = \frac{2}{\pi r}\int_0^{\pi/2}\sin^2\gamma\, d\gamma \int_0^{\pi/2}\left[1 - \exp\left(-2r\frac{\cos\varepsilon}{\sin\gamma}\right)\right] \times \left[1 - \exp\left(-r\frac{\sqrt{\delta^2 - \sin^2\varepsilon} - \cos\varepsilon}{\sin\gamma}\right)\right]\cos\varepsilon\, d\varepsilon,$$

$$\varphi_{F_2F_1} = \frac{4}{\pi}\int_0^{\pi/2}\sin^2\gamma\, d\gamma \int_0^{\pi/2} \times \exp\left(-r\frac{\sqrt{\delta^2 - \sin^2\varepsilon} - \cos\varepsilon}{\sin\gamma}\right)\cos\varepsilon\, d\varepsilon.$$

It is advisable here to include only certain values of these quantities as control points for the programs (Tables 1, 2, and 3). The quantities $\varphi_{ij}k_j$ and $\varphi_{F_2F_1}$ have been derived by means of Gaussian quadratures with five abscissas.

Table 2

Angle factors multiplied by the attenuation coefficient from an infinite cylinder (dimensionless radius r) to an adjacent coaxial cylindrical layer (dimensionless radius δr).

| r | δ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1.05 | 1.1 | 1.2 | 1.5 | 2 | 3 | ∞ |
| 0.1 | 0079 | 0154 | 0296 | 0683 | 1253 | 2228 | 8847 |
| 0.5 | 0247 | 0468 | 0868 | 1831 | 2974 | 4335 | 5960 |
| 1 | 0325 | 0605 | 1074 | 2064 | 2988 | 3751 | 4071 |
| 2 | 0362 | 0644 | 1066 | 1763 | 2179 | 2342 | 2364 |
| 5 | 0333 | 0533 | 0756 | 0958 | 0989 | 0992 | 0992 |

Table 3

Angle factors for coaxial infinite cylindrical surfaces—from the inside to the outside. The dimensionless radii are r and δr.

| r | δ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1.05 | 1.1 | 1.2 | 1.5 | 2 | 2.5 | 3 |
| 0.1 | 99117 | 98313 | 96815 | 92775 | 86808 | 81419 | 76462 |
| 0.5 | 95698 | 91936 | 85290 | 69397 | 50511 | 37312 | 27789 |
| 1 | 91652 | 84699 | 73128 | 48966 | 26501 | 14777 | 08369 |
| 2 | 84224 | 72228 | 54358 | 25137 | 07726 | 02507 | 00836 |
| 5 | 66099 | 45992 | 23643 | 03879 | 00238 | 00016 | 00003 |

NOTATION

$a_i$ is the attenuating ability of zone i in a certain direction; $\tau_i = \int_{l_i} k dl$ is the dimensionless i-th section of attenuation of a ray; $l_i$ is its length, m; $k = \alpha + \beta$ is the attenuation factor, $m^{-1}$; $\alpha$ and $\beta$ are the absorption and scattering coefficients, $m^{-1}$; $B = \sigma T^4/\pi$; $\sigma = 5.58 \cdot 10^{-8}$ W/m$^2$ deg; T is the temperature, °K; I is the radiation intensity, W/m$^2$ steradian; q is the density of a hemispherical flow, W/m$^2$; $\theta$ is the angle between the normal to a surface element dF and a ray; $d\omega$ is the element of a solid angle limiting a beam of rays; $\xi$ and $\varphi$ are the local and mean angle factors; V and F are the volume and surface; M and $\mu$ are the set and density of the set of rays, connecting elements of two zones; r and R are the dimensionless radii (radii multiplied by attenuation factor; see Fig. 2); x is a dimensionless argument; $\varepsilon$ is the angle between the normal to surface element $dF_2$ and ray projection onto the section of a two-dimensional body; $\pi/2 - \gamma$ is the angle between this projection and the ray so that $\cos\theta = \sin\gamma \cos\varepsilon$.

REFERENCES

1. S. P. Detkov, Izv. AN SSSR, Energetika i transport, no. 6, 1966.

2. A. S. Nevskii, IFZh [Journal of Engineering Physics], 8, no. 5, 613, 1965.

3. S. P. Detkov, Teplofizika vysokikh temperatur, **2**, no. 1, 82, 1964.

4. A. S. Nevskii, Radiative Heat Exchange in Metallurgical Furnaces and Boilers [in Russian], Metallurgizdat, 1958.

5. A. S. Nevskii, collection: Scientific works of the All-Union Scientific-Research Institute for Metallurgy and Thermal Physics. The Thermal Work of Metallurgical Furnaces [in Russian], Sverdlovsk, 1965.

6. J. C. Y. Koh, Int. J. Heat Mass Transfer, 373, 1965.

7. K. Ya. Kondrat'ev, Radiative Heat Exchange in the Atmosphere, Leningrad, 1956.

8. I. R. Mikk, Teplofizika vysokikh temperatur, **1**, no. 1, 128, 1963; "Heat transfer and furnace processes," Sb. trudov Tallinskogo politekhnicheskogo in-ta, Tallin, seriya A, no. 206, p. 3, 1963.

9. S. S. R. Murty, Int. J. Heat Mass Transfer, 8, 1203, 1965.

26 January 1967

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